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Stability for delay-differential equations with N delays

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Consider the delay-differential equations

$$x'(t)=\sum_{i=1}^{N}F_i(t,x_t), \tag{DDE}$$

where $F_i\in C(\boldsymbol{R}^+\times C^{q_i}(H),\boldsymbol{R})$, $\boldsymbol{R}^+=[0,\infty)$, $q_i>0$, $H>0$, $C^{q_i}(H)=\{\phi\in C^{q_i}:\|\phi\|<H\}$, $C^{q_i}=\{\phi:[-q_i,0]\longrightarrow\boldsymbol{R}: continuous\}$ and $x_t(s)=x(t+s)$ for $s\in[-q_i,0]$. We suppose that $q_1\leq q_2\leq\cdots\leq q_N$.

Definition 1 (Yorke condition) *We say that a continuous function $F\in C(\boldsymbol{R}^+\times C^q(H),\boldsymbol{R})$ satisfies a Yorke condition for $q>0$, if there exists $\alpha\geq 0$ such that*

$$-\alpha M_q(\phi)\leq F(t,\phi)\leq\alpha M_q(-\phi),$$

for all $t\geq 0$ and $\phi\in C^q(H)$, where $M_q(\phi)=\max\{0,\sup_{s\in[-q,0]}\phi(s)\}$.

For a continuous function $F\in C(\boldsymbol{R}^+\times C^q(H),\boldsymbol{R})$, there exists a continuous function $a\in C(\boldsymbol{R}^+,\boldsymbol{R}^+)$ such that

$$-a(t)M_q(\phi)\leq F(t,\phi)\leq a(t)M_q(-\phi), \tag{C1}$$

for all $t\geq 0$ and $\varphi\in C^q(H)$.

For a continuous function $F\in C(\boldsymbol{R}^+\times C^q(H),\boldsymbol{R})$, there exists a continuous function $a\in C(\boldsymbol{R}^+,\boldsymbol{R}^+)$ such that

$$-a(t)\sup_{s\in[-q,0]}\phi(s)\leq F(t,\phi)\leq a(t)\sup_{s\in[-q,0]}(-\phi(s)), \tag{C2}$$

for all $t\geq 0$ and $\phi\in C^q(H)$.

Remark. If a continuous function $F\in C(\boldsymbol{R}^+\times C^q(H),\boldsymbol{R})$ satisfies one of the above conditions, then $F(t,0)\equiv 0$ for all $t\geq 0$.

Theorem 1 *Suppose that for $i=1,\cdots,N$, F_i satisfies a Yorke condition for $q_i>0$.*
(1) *If one of the following conditions is satisfied,*

(i) $A<\dfrac{1}{q_N}$,

(ii) $\dfrac{1}{q_{N-k+1}}\leq A\left(<\dfrac{1}{q_{N-k}}\right)$ *and* $\dfrac{1}{2A}\sum_{i=1}^{N-k}\alpha_i(Aq_i-1)^2+\Lambda\leq\dfrac{3}{2}$ *for* $k=1,\cdots,N-1$,

(iii) $A > \dfrac{1}{q_1}$ *and* $\Lambda \leq \dfrac{3}{2}$,

then the zero solution of (DDE) is uniformly stable.
(2) *If one of the following conditions is satisfied,*

(i) $A < \dfrac{1}{q_N}$,

(ii) $\dfrac{1}{q_{N-k+1}} \leq A \left(< \dfrac{1}{q_{N-k}} \right)$ *and* $\dfrac{1}{2A} \sum_{i=1}^{N-k} \alpha_i (Aq_i - 1)^2 + \Lambda < \dfrac{3}{2}$ *for* $k = 1, \cdots, N-1$,

(iii) $A > \dfrac{1}{q_1}$ *and* $\Lambda < \dfrac{3}{2}$,

then $\lim_{t \to \infty} x(t; t_0, \phi)$ *exists for any* $t_0 \geq 0$ *and* $\phi \in C^{q_N}(He^{-2A\lambda_N})$,
where $A = \sum_{i=1}^{N} \alpha_i$ *and* $\Lambda = \sum_{i=1}^{N} \alpha_i q_i$.

Theorem 2 *Suppose that for* $i = 1, \cdots, N$, F_i *satisfies a condition* $(C1)$ *for* $q_i > 0$, *that is, there exists a continuous function* $a_i \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that*

$$-a_i(t) M_{q_i}(\phi) \leq F_i(t, \phi) \leq a_i(t) M_{q_i}(-\phi),$$

for all $t \geq 0$ *and* $\phi \in C^{q_i}(H)$. *Moreover we suppose that for* $i = 1, \cdots, N$, *there exist* $\alpha_i \geq 0$ *and a continuous function* $a \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that* $a_i(t) \leq \alpha_i a(t)$ *for all* $t \geq 0$.
(1) *If the following condition is satisfied,*

$$\frac{1}{2A} \sum_{i=1}^{N-1} \alpha_i (A\lambda_i - 1)^2 + \Lambda \leq \frac{3}{2},$$

then the zero solution of (DDE) is uniformly stable.
(2) *If the following condition is satisfied,*

$$\frac{1}{2A} \sum_{i=1}^{N-1} \alpha_i (A\lambda_i - 1)^2 + \Lambda < \frac{3}{2},$$

then $\lim_{t \to \infty} x(t; t_0, \phi)$ *exists for any* $t_0 \geq 0$ *and* $\phi \in C^{q_N}(He^{-2A\lambda_N})$,
where $A = \sum_{i=1}^{N} \alpha_i$, $\Lambda = \sum_{i=1}^{N} \alpha_i \lambda_i$ *and* $\lambda_i = \sup_{t \geq 0} \int_t^{t+q_i} a(s) ds$.

Corollary 1 *Suppose that the conditions in Theorem 2 are satisfied. If the following conditions are satisfied:*

$$\mu \geq 1 \textit{ and } \Lambda = \sum_{i=1}^{N} \alpha_i \lambda_i \leq \frac{3}{2},$$

then the zero solution of (DDE) is uniformly stable, where $\mu = \inf_{t \geq 0} \int_t^{t+q_1} A(s)ds$ *and* $A(t) = \sum_{i=1}^{N} a_i(t)$.

Remark. In the case of $N = 1$ in Corollary 1, it is the same as the 3/2 *Stability Theorem* proved by [2].

Theorem 3 *Suppose that for* $i = 1, \cdots, N$, F_i *satisfies a condition* $(C2)$ *for* $q_i > 0$, *that is, there exist* $a_i \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that*

$$-a_i(t) \sup_{s \in [-q_i, 0]} \phi(s) \leq F_i(t, \phi) \leq a_i(t) \sup_{[-q_i, 0]} (-\phi(s))$$

for all $t \geq 0$ *and* $\phi \in C^{q_i}(H)$. *Moreover we suppose that for* $i = 1, \cdots, N$, *there exist* $\alpha_i \geq 0$ *and a continuous function* $a \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that* $a_i(t) \leq \alpha_i a(t)$ *for all* $t \geq 0$. *If the following conditions are satisfied,*

$$\frac{1}{2A} \sum_{i=1}^{N-1} \alpha_i (A\lambda_i - 1)^2 + \Lambda < \frac{3}{2} \quad \textit{and} \quad \int_0^{\infty} A(t)dt = \infty,$$

then the zero solution of (DDE) is asymptotically stable, where $A = \sum_{i=1}^{N} \alpha_i$, $A(t) = \sum_{i=1}^{N} a_i(t)$, $\Lambda = \sum_{i=1}^{N} \alpha_i \lambda_i$ *and* $\lambda_i = \sup_{t \geq 0} \int_t^{t+q_i} a(s)ds$.

Theorem 4 *Suppose that for* $i = 1, \cdots, N$, F_i *satisfies a condition* $(C1)$ *for* $q_i > 0$, *that is, there exist* $a_i \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that*

$$-a_i(t) M_{q_i}(\phi) \leq F_i(t, \phi) \leq a_i(t) M_{q_i}(-\phi),$$

for all $t \geq 0$ *and* $\phi \in C^{q_i}(H)$. *Moreover we suppose that for for* $i = 1, \cdots, N$, *there exist* $\alpha_i \geq 0$ *and a continous function* $a \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that* $a_i(t) \leq \alpha_i a(t)$ *for all* $t \geq 0$ *and that for all sequences* $\{t_n\} \nearrow \infty$ *and* $\phi_n \in C^{q_i}(H)$ *converging to a nonzero constant function in* $C^{q_i}(H)$, $\sum_{i=1}^{N} F_i(t_n, \phi_n)$ *does not converge to* 0. *If the following conditions are satisfied,*

$$\frac{1}{2A} \sum_{i=1}^{N-1} \alpha_i (A\lambda_i - 1)^2 + \Lambda < \frac{3}{2} \quad \textit{and} \quad \mu = \inf_{t \geq 0} \int_t^{t+q_1} A(s)ds > 0,$$

then the zero solution of (DDE) is uniformly asymptotically stable, where $A = \sum_{i=1}^{N} \alpha_i$, $A(t) = \sum_{i=1}^{N} a_i(t)$, $\Lambda = \sum_{i=1}^{N} \alpha_i \lambda_i$ *and* $\lambda_i = \sup_{t \geq 0} \int_t^{t+q_i} a(s)ds$.

Theorem 5 *Suppose that for* $i = 1, \cdots, N$, F_i *satisfies a condition (C2) for* $q_i > 0$, *that is, there exist* $a_i \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that*

$$-a_i(t) \sup_{[-q_i,0]} \phi(s) \leq F_i(t, \phi) \leq a_i(t) \sup_{[-q_i,0]} (-\phi(s)),$$

for all $t \geq 0$ *and* $\phi \in C^{q_i}(H)$. *Moreover we suppose that for* $i = 1, \cdots, N$, *there exist* $\alpha_i \geq 0$ *and a continous function* $a \in C(\boldsymbol{R}^+, \boldsymbol{R}^+)$ *such that* $a_i(t) \leq \alpha_i a(t)$ *for all* $t \geq 0$. *If the following conditions are satisfied,*

$$\frac{1}{2A} \sum_{i=1}^{N-1} \alpha_i (A\lambda_i - 1)^2 + \Lambda < \frac{3}{2} \quad \text{and} \quad \mu = \inf_{t \geq 0} \int_t^{t+q_1} A(s)ds > 0,$$

then the zero solution of (DDE) is uniformly asymptotically stable, where $A = \sum_{i=1}^{N} \alpha_i$, $A(t) = \sum_{i=1}^{N} a_i(t)$, $\Lambda = \sum_{i=1}^{N} \alpha_i \lambda_i$ *and* $\lambda_i = \sup_{t \geq 0} \int_t^{t+q_i} a(s)ds$.

Above theorems are proved by the following two lemmas. Give the proof of two lemmas.

Lemma 1 *For some* $t_1 \geq 0$, *let* $x(t)$ *be a solution of (DDE) on* $[t_1 - q_N, t_1]$ *such that* $|x(t)| > 0$ *for all* $t \in (t_1 - q_N, t_1)$, *then*

$$x(t_1)x'(t_1) \leq 0.$$

Proof. If $x(t) > 0$ for all $t \in (t_1 - q_N, t_1)$, then $F_i(t_1, x_{t_1}) \leq a_i(t) M_{q_i}(-x_{t_1}) = 0$ for $i = 1, \cdots, N$. Therefore

$$x(t_1)x'(t_1) = \sum_{i=1}^{N} x(t_1) F_i(t_1, x_{t_1}) \leq 0 .$$

Similarly, if $x(t) < 0$ for all $t \in (t_1 - q_N, t_1)$, then $F_i(t_1, x_{t_1}) \geq -a_i(t) M_{q_i}(x_{t_1}) = 0$ for $i = 1, \cdots, N$, and hence $x(t_1)x'(t_1) \leq 0$, so this lemma is proved.

Lemma 2 *Suppose that the conditions in Theorem 2 are satisfied. Let* $x(t)$ *be a solution of (DDE) on* $[t_1 - q_N, t_1]$ *such that* $T > t_1 + q_N$ *and* $x(t_1) = 0$, *then*

$$|x(t)| \leq \theta \sup_{s \in [t_1 - 2q_N, t_1]} |x(s)|$$

for all $t \in [t_1, T]$, *where* $\theta = \max\left\{1 - \left(\frac{3}{2} - \Lambda\right)\mu, \frac{1}{2A}\sum_{i=1}^{N-1}\alpha_i(A\lambda_i - 1)^2 + \Lambda - \frac{1}{2}\right\}$ *and* $\mu = \inf_{t\geq 0}\int_t^{t+q_1} A(s)ds$.

Proof. Suppose not. Let $r_0 = \sup_{s\in[t_1-2q_N,t_1]}|x(s)|$, $t_3 = \inf\{t > t_1; |x(t)| > \theta r_0\}$ and $t_2 = \sup\{t < t_3; x(t) = 0\}$. Then $|x(t_3)| = \theta r_0$ and $|x(t)| > 0$ for all $t \in (t_2, t_3]$. We suppose that $x(t) > 0$ for all $t \in (t_2, t_3]$, since the proof is similar in the other case. Then from the definition of t_3, there exists $t_4 \geq t_3$ such that $x'(t_4) > 0$, $x(t) > 0$ for all $t \in (t_3, t_4]$ and

$$x(t_4) = \sup_{s\in[t_3,t_4]} x(s) \ . \tag{1}$$

It follows from Lemma3.1 that

$$t_4 < t_2 + q_N \ . \tag{2}$$

It is easy to see that

$$x(t_4) \geq \theta \sup_{s\in[t_1-2q_N,t_4]} |x(t_4)|. \tag{3}$$

Let $r = \sup_{s\in[t_3,t_4]} x(s)$, then by (C1)

$$|x'(t)| \leq \sum_{i=1}^{N} a_i(t) \sup_{s\in[t-q_i,t]} |x(s)| \leq r\left(\sum_{i=1}^{N} a_i(t)\right) \leq rAa(t)$$

for all $t \in [t_1 - q_N, t_4]$, and hence

$$|x(t)| = |x(t_2) - x(t)| = \left|\int_t^{t_2} x'(s)ds\right| \leq rA\left|\int_t^{t_2} a(s)ds\right| \tag{4}$$

for all $t \in [t_1 - q_N, t_4]$. Moreover it follows from (C1) and (4) that for $s \in [0, \min\{q_1, t_4 - t_2\}]$

$$\begin{aligned}
x'(t_2+s) &= \sum_{i=1}^{N} F_i(t_2+s, x_{t_2+s}) \\
&\leq \sum_{i=1}^{N} a_i(t_2+s) \sup_{u\in[t_2+s-q_i,t_2+s]} (-x(u)) \\
&\leq \sum_{i=1}^{N} \alpha_i a(t_2+s) \sup_{u\in[t_2+s-q_i,t_2]} |x(u)| \\
&\leq r\sum_{i=1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{t_2+s-q_i}^{t_2} a(s)ds\right\} \\
&\leq r\sum_{i=1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\}.
\end{aligned}$$

Let $m = \sup\{k; t_2 + q_k < t_4, 1 \le k \le N-1\}$, then for $s \in (q_1, q_2]$

$$\begin{aligned} x'(t_2+s) &= \sum_{i=1}^{N} F_i(t_2+s, x_{t_2+s}) \\ &\le \sum_{i=2}^{N} F_i(t_2+s, x_{t_2+s}) \\ &\le \sum_{i=2}^{N} a_i(t_2+s) \sup_{u\in[t_2+s-q_i, t_2+s]} (-x(u)) \\ &\le \sum_{i=2}^{N} a_i(t_2+s) \sup_{u\in[t_2+s-q_i, t_2]} |x(u)| \\ &\le r\sum_{i=2}^{N} \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\}. \end{aligned}$$

Similarly, for $s \in (q_k, q_{k+1}]$

$$x'(t_2+s) \le r \sum_{i=k+1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\}$$

for $k = 2, \cdots, m-1$. For $s \in (q_m, t_4 - t_2]$

$$x'(t_2+s) \le r \sum_{i=m+1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\}.$$

Therefore,

$$\begin{aligned} x(t_4) &= x(t_4) - x(t_2) = \int_0^{t_4-t_2} x'(t_2+s)ds \\ &\le r\int_0^{q_1} \sum_{i=1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\} \\ &\quad + r\sum_{k=1}^{m-1}\int_{q_k}^{q_{k+1}} \sum_{i=k+1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\} \\ &\quad + r\int_{q_m}^{t_4-t_2} \sum_{i=m+1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\} \\ &< r\sum_{i=1}^{m}\int_0^{q_i} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\} \\ &\quad + r\int_{q_m}^{q_{m+1}} \sum_{i=m+1}^{N} \alpha_i \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\} \\ &\le r\sum_{i=1}^{N} \alpha_i \int_0^{q_i} \min\left\{a(t_2+s), a(t_2+s)A\int_{s-q_i}^{0} a(t_2+s)ds\right\} \end{aligned}$$

By above calculation, we have

$$x(t_4) < r\sum_{i=1}^{N}\alpha_i\int_0^{q_i}\min\left\{a(t_2+s),a(t_2+s)A\int_{s-q_i}^{0}a(t_2+s)ds\right\} \tag{5}$$

Let $\gamma_i = A\int_{-q_i}^{0}a(t_2+s)ds$ for $i=1,\cdots,N$. Now we discuss the following cases:

(i) $\gamma_N < 1$.

Then (5) yields

$$\begin{aligned}
x(t_4) &< r\sum_{i=1}^{N}\alpha_i\int_0^{q_i}a(t_2+s)A\int_{s-q_i}^{0}a(t_2+u)duds \\
&= rA\sum_{i=1}^{N}\alpha_i\int_{-q_i}^{0}a(t_2+u)\int_u^{u+q_i}a(t_2+s)dsdu \\
&\quad -rA\sum_{i=1}^{N}\alpha_i\int_{-q_i}^{0}a(t_2+u)\int_u^{0}a(t_2+s)dsdu \\
&= r\sum_{i=1}^{N}\alpha_i\lambda_i\gamma_i+\frac{r}{2}A\sum_{i=1}^{N}\alpha_i\int_{-q_i}^{0}\frac{d}{du}\left(\int_u^0 a(t_2+s)ds\right)^2du \\
&= r\sum_{i=1}^{N}\alpha_i\lambda_i\gamma_i-\frac{r}{2}A\sum_{i=1}^{N}\alpha_i\left(\int_{-q_i}^{0}a(t_2+s)ds\right)^2 \\
&= r\sum_{i=1}^{N}\frac{\alpha_i}{A}\left\{-\frac{1}{2}(1-\gamma_i)(2-\gamma_i)+1-\left(\frac{3}{2}-A\lambda_i\right)\gamma_i\right\} \\
&\le r\sum_{i=1}^{N}\frac{\alpha_i}{A}\left\{1-\left(\frac{3}{2}-A\lambda_i\right)\mu\right\} \\
&= r\left\{1-\left(\frac{3}{2}-\Lambda\right)\mu\right\} \\
&\le \theta r
\end{aligned}$$

which is a contradiction for (3).

(ii) Case k for $k=1,\cdots,N-1$.

Suppose $\gamma_{N-k} < 1$ and $\gamma_{N-k+1} \ge 1$. Then there exist $\tilde{q}_i \le q_i$ such that

$A\int_{\tilde{q}_i-q_i}^{0} a(t_2+s)ds = 1$ for $i = N-k+1,\cdots,N$. Thus we have

$$\begin{aligned}
x(t_4) &< r\sum_{i=1}^{N-k}\alpha_i\int_0^{q_i} a(t_2+s)A\int_{s-q_i}^{0} a(t_2+u)duds \\
&\quad + r\sum_{i=N-k+1}^{N}\alpha_i\int_0^{\tilde{q}_i} a(t_2+s)ds + r\sum_{i=N-k+1}^{N}\alpha_i\int_{\tilde{q}_i}^{q_i} a(t_2+s)A\int_{s-q_i}^{0} a(t_2+u)duds \\
&= rA\sum_{i=1}^{N-k}\alpha_i\int_{-q_i}^{0} a(t_2+u)\int_u^{u+q_i} a(t_2+s)dsdu \\
&\quad - rA\sum_{i=1}^{N-k}\alpha_i\int_{-q_i}^{0} a(t_2+u)\int_u^{0} a(t_2+s)dsdu \\
&\quad + rA\sum_{i=N-k+1}^{N}\alpha_i\int_{\tilde{q}_i-q_i}^{0} a(t_2+u)\int_0^{\tilde{q}_i} a(t_2+s)dsdu \\
&\quad + rA\sum_{i=N-k+1}^{N}\alpha_i\int_{\tilde{q}_i-q_i}^{0} a(t_2+u)\int_{\tilde{q}_i}^{u+q_i} a(t_2+s)dsdu \\
&\le rA\sum_{i=1}^{N-k}\alpha_i\lambda_i\int_{-q_i}^{0} a(t_2+u)du - \frac{r}{2}A\sum_{i=1}^{N-k}\alpha_i\left(\int_{-q_i}^{0} a(t_2+u)du\right)^2 \\
&\quad + rA\sum_{i=N-k+1}^{N}\alpha_i\int_{\tilde{q}_i-q_i}^{0} a(t_2+u)\int_u^{u+q_i} a(t_2+s)dsdu \\
&\quad - rA\sum_{i=N-k+1}^{N}\alpha_i\int_{\tilde{q}_i-q_i}^{0} a(t_2+u)\int_u^{0} a(t_2+s)dsdu \\
&= r\sum_{i=1}^{N-k}\alpha_i\lambda_i\gamma_i - \frac{r}{2A}\sum_{i=1}^{N-k}\alpha_i\gamma_i^2 + rA\sum_{i=N-k+1}^{N}\alpha_i\lambda_i\int_{\tilde{q}_i-q_i}^{0} a(t_2+u)du \\
&\quad - \frac{r}{2}A\sum_{i=N-k+1}^{N}\alpha_i\left(\int_{\tilde{q}_i-q_i}^{0} a(t_2+u)du\right)^2 \\
&= r\sum_{i=1}^{N-k}\alpha_i\lambda_i\gamma_i - \frac{r}{2A}\sum_{i=1}^{N-k}\alpha_i\gamma_i^2 + r\sum_{i=N-k+1}^{N}\alpha_i\lambda_i - \frac{r}{2A}\sum_{i=N-k+1}^{N}\alpha_i \\
&= r\sum_{i=1}^{N-k}\alpha_i\lambda_i\gamma_i - \frac{r}{2A}\sum_{i=1}^{N-k}\alpha_i\gamma_i^2 + r\sum_{i=1}^{N}\alpha_i\lambda_i - r\sum_{i=1}^{N-k}\alpha_i\lambda_i - \frac{r}{2A}\sum_{i=1}^{N}\alpha_i + \frac{r}{2A}\sum_{i=1}^{N-k}\alpha_i \\
&= \frac{r}{2A}\sum_{i=1}^{N-k}\alpha_i(1-2A\lambda_i+2A\lambda_i\gamma_i-\gamma_i^2) + r\sum_{i=1}^{N}\alpha_i\lambda_i - \frac{r}{2} \\
&\le \frac{r}{2A}\sum_{i=1}^{N-k}\alpha_i(A\lambda_i-1)^2 + r\sum_{i=1}^{N}\alpha_i\lambda_i - \frac{r}{2} \\
&\le \theta r.
\end{aligned}$$

We have a contradiction for (3).

(iii) $\gamma_1 \geq 1$.

Then there exist $\tilde{q}_i \leq q_i$ such that $A\int_{\tilde{q}_i - q_i}^{0} a(t_2 + u)du = 1$ for $i = 1, \cdots, N$. Similary, we have

$$x(t_4) \ < \ \sum_{i=1}^{N} \alpha_i \lambda_i - \frac{r}{2} \leq \theta r.$$

We have also a contradiction in this case. Thus, the proof is now complete.

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